Pascal-like triangles and Fibonacci-like sequences

1. Introduction and mathematical background

In [1], one of the authors of this article demonstrated with three others how Pascal-like triangles arose from the probabilities associated with the various outcomes of a game of Definition 1 with the condition that $s = 1$. They also showed how Fibonacci-like sequences arose from Pascal-like triangles, and demonstrate the existence of simple relationships between these Fibonacci-like sequences and the Fibonacci sequence itself. In this article, the authors generalize the result of [1], and they show that Pascal-like triangles arise also from a game of Definition 1 for an arbitrary natural number p, and Fibonacci-like sequences arise from these Pascal-like triangles. The following Definition 1 is the same as Definition 1 in [1] when $s = 1$.

Definition 1. Let p, n, m and s be fixed positive integers, with $m \leq n$. There are p players $\Theta_1, \Theta_2, ..., \Theta_p$ seated around a circular table, and the game starts with player Θ_1. Proceeding in order, a box containing n identically-sized cards is passed from hand to hand. All of these cards are white except for m of them, which are red. When a player receives the box he or she draws out a card at random (i.e. the player cannot see inside the box) s times, and these cards are not returned to the box. In this way, Player Θ_1 draws a card in the first round, the second round,...,s th round. We call this group of rounds Player Θ_1's turn. Then, Player Θ_2 draws a card in the $s+1$ th,...,$2s$ th round. We call this group of rounds Player Θ_2's turn. The game continues and Player Θ_p draws a card in the $(p-1)s+1$ th,...,$(p-1)s+p = ps$ th round. Then we call the group of Player Θ_1's turn, Player Θ_2's turn,...,Player Θ_p's turn as the first period. Next, Player Θ_1 draws a card again, and the game contiues. In other words, a player's turn consists of s rounds, and a period consists of p turns. The first player to draw out a red card loses the game, and the game ends at this round.

In the remainder of this section, the authors present some results of [1], and they use Definition 1 for $s = 1$. The following definition is the same as Definition 2 of [1].

Definition 2. Let $U(p, n, m, v) = \sum_{z=0}^{t-1} {}_{n-v-pz}C_{m-1}$, where $t = \lfloor \frac{n-m+p-v+1}{p} \rfloor$.

Lemma 1. $U(p, n, m, v)$ is the possible arrangements of positions of the m red cards that would lead to Player Θ_v losing the game when $s = 1$.

This is from Definition 2 and the comment following Definition 2 in [1].

Definition 3. Let $f(p, n, m, v)$ be the probability that the v th player loses in the game of Definition 1 when $s = 1$.

Theorem 1. $f(p, n, m, v) = \frac{U(p,n,m,v)}{{}_nC_m}$.

This is Theorem 1 of [1].

Theorem 2. For any positive integers n, m, p and v such that $m \leq n$ and $v \leq p$, $U(p, n, m, v) + U(p, n, m+1, v) = U(p, n+1, m+1, v)$.

This is Theorem 2 of [1].

Remark 1. By Theorem 1 and Theorem 2, the set $\{f(p, n, m, v) : m \leq n, n = 1, 2, ...\}$ has a pattern similar to Pascal's triangle for fixed positive integers p and v.

Example 1. Here, we assume that $p = 2$ and $v = 1$. As an illustrative example for Remark 1, the Pascal-like triangle formed from $\{f(2, n, m, 1), 1 \leq m \leq n, n = 1, 2, ..., 6, 7\}$ is shown in Figure 1. Clearly, the triangle in Figure 1 has an elegant property. For example, see $f(2, 6, 2, 1) = \frac{9}{15}, f(2, 6, 3, 1) = \frac{13}{20}$, $f(2, 7, 3, 1) = \frac{22}{35}$. Note that $9 + 10 = 16$ and $15 + 20 = 35$. As you see in Figure 1, the denominators and numerators of the fractions form Pascal-like triangles.

Figure 1.

$$
\begin{array}{c}
\frac{1}{1} \\
\frac{1}{2}, \frac{1}{1} \\
\frac{2}{3}, \frac{2}{3}, \frac{1}{1} \\
\frac{2}{4}, \frac{4}{6}, \frac{3}{4}, \frac{1}{1} \\
\frac{3}{5}, \frac{6}{10}, \frac{7}{10}, \frac{4}{5}, \frac{1}{1} \\
\frac{3}{6}, \frac{\mathbf{9}}{\mathbf{15}}, \frac{\mathbf{13}}{\mathbf{20}}, \frac{11}{15}, \frac{5}{6}, \frac{1}{1} \\
\frac{4}{7}, \frac{12}{21}, \frac{\mathbf{22}}{\mathbf{35}}, \frac{24}{35}, \frac{16}{21}, \frac{6}{7}, \frac{1}{1}
\end{array}
$$

Figure 2.

1
1,1
2,2,1
2,4,3,1
3,6,7,4,1
3,9,13,11,5,1
4,12,22,24,16,6,1

Figure 3.

1
1,1
2,2,**1**
2,**4**,3,1
3,6,7,4,1
3,9,13,11,5,1
4,12,22,24,16,6,1

Numbers in Figure 2 and Figure 3 are the numerators of the fractions in Figure 1.

It is well known that the numbers on diagonals of the Pascal's triangle add to the Fibonacci sequence, but the numbers on diagonals of the triangle in Figure 2 add to Fibonacci like sequences. Let b_n be the sequence that is made in this way. Then, $b_1 = 1, b_2 = 1, b_3 = 2 + 1 = 3, b_4 = 2 + 2 = 4, b_5 = 3 + 4 + 1 = 8, b_6 = 3 + 6 + 3 = 12, b_7 = 4 + 9 + 7 + 1 = 21, \cdots$. In Figure 3 we demonstrate how we added the numbers in the triangle to make $b_5 = 3 + 4 + 1 = 8$. These numbers are printed in bold letter.

The definition of $b_n, n = 1, 2, 3, ...$ is given in (1.1).

$$b_n = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} U(2, n-k, k+1, 1). \tag{1.1}$$

It is easy to see that the rule of this sequence is

$$b_n = b_{n-1} + b_{n-2} + \begin{cases} 1 & (n = 1 \pmod 2) \\ 0 & (n \neq 1 \pmod 2)\,. \end{cases} \tag{1.2}$$

2. Generalized Games That Produce Pascal-like Triangle

In this section, we generalize the result of [1] presented in Section 1. The game of Definition 1 for $s = 1$ is mathematically the same as a Russian roulette game in which p players take turns and shoot themselves. To calculate the probability of the game of Definition 1 for an arbitrary natural number s, it is also easier to use the data structure of Russian roulette. We suppose that cards are arranged in a cylinder-like component as the Figure 4 into which n cards are placed. First, the card on the far left is to be picked up, and the last card to be picked up is on the far right.

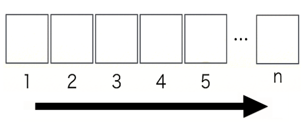

Figure 4. cards on n squares

The following Lemma 2 presents a well known formula, and we use this very often throughout this article.

Lemma 2.

$${}_nC_m + {}_nC_{m+1} = {}_{n+1}C_{m+1}. \tag{2.1}$$

Definition 4. We denote by $R(n, m, y)$ the number of combinations of positions of red cards and white cards when the game ends in the y th round. Note that $R(n, m, y)$ is independent of p and s, where p is the number of players and s is the number of times a player draws a card in his or her turn.

Lemma 3. For any natural number n, m, y such that $m \leq n$ and $y \leq n - m + 1$, $R(n, m, y) = {}_{n-y}C_{m-1}$.

Proof. The game ends in the y th round if a red card is in the y th place and other $m - 1$ red cards are positioned after the yth place. There are ${}_{n-y}C_{m-1}$ ways to arranging the cards in this way so ${}_{n-y}C_{m-1}$ gives us the number of ways that the game can end in the y th round. $\square$

Example 2. We calculate $R(6, 3, 3)$ that is the number of combinations of positions of three red cards and three white cards when the game ends in the 3 rd round. The game ends in the 3 th round if a card with red number is in the 3 th round and other two red cards are positioned after the 3th place. There are ${}_3C_2$ ways to put cards with red cards into places this way. Therefore $R(6, 3, 3) = {}_3C_2$.

Definition 5. We denote by $U(p,n,m,s,v)$ the number of combinations of positions of red cards that Θ_v (the $v-$th player) loses the game of Definition 1.

Theorem 3.

$$U(p,n,m,s,v)=\sum_{h=1}^{s}\sum_{i=1}^{\lfloor\frac{n-m-s(v-1)-h+1+ps}{ps}\rfloor}{}_{n-(i-1)ps-s(v-1)-h}C_{m-1}. \tag{2.2}$$

Proof. First, the v th player Θ_v draws a card at $s(v-1)+1$ th round, $s(v-1)+2$ th round, ...,$s(v-1)+s = sv$ th round in the first period. The game continues in this way, and the v th player Θ_v draws a card at $(i-1)ps+s(v-1)+1$ th round, $(i-1)ps+s(v-1)+2$ th round, ...,$(i-1)ps+s(v-1)+s=(i-1)ps+sv$ th round in the i th period.

For a natural number h such that $1 \leq h \leq s$, the $v-$th player can lose the game at $(i-1)ps+s(v-1)+h$ th round if a red card is in $(i-1)ps+s(v-1)+h$ th round and other $m-1$ red cards are positioned after the $(i-1)ps+s(v-1)+h$ th place. Then,

$$n-(i-1)ps-s(v-1)-h \geq m-1 \tag{2.3}$$

, and there are

$${}_{n-(i-1)ps-s(v-1)-h}C_{m-1} \tag{2.4}$$

ways to put cards into places this way.
By (2.3)

$$1 \leq i \leq \lfloor\frac{n-m-s(v-1)-h+1+ps}{ps}\rfloor. \tag{2.5}$$

By (2.4) and (2.5) we have (2.2). □

Lemma 4.

$$U(p,n+1,m+1,s,v)=U(p,n,m+1,s,v)+U(p,n,m,s,v).$$

Proof. By Theorem 3

$$U(p,n+1,m+1,s,v)=\sum_{h=1}^{s}\sum_{i=1}^{\lfloor\frac{n-m-s(v-1)-h+1+ps}{ps}\rfloor}{}_{n+1-(i-1)ps-s(v-1)-h}C_{m}, \tag{2.6}$$

$$U(p,n,m+1,s,v)=\sum_{h=1}^{s}\sum_{i=1}^{\lfloor\frac{n-m-s(v-1)-h+ps}{ps}\rfloor}{}_{n-(i-1)ps-s(v-1)-h}C_{m}. \tag{2.7}$$

and

$$U(p,n,m,s,v)=\sum_{h=1}^{s}\sum_{i=1}^{\lfloor\frac{n-m-s(v-1)-h+1+ps}{ps}\rfloor}{}_{n-(i-1)ps-s(v-1)-h}C_{m-1}. \tag{2.8}$$

We fix h. If we prove that (2.9) is equal to the sum of (2.10) and (2.11), (2.6) is equal to the sum of (2.7) and (2.8). Then the proof of this lemma is finished.

$$\sum_{i=1}^{\lfloor\frac{n-m-s(v-1)-h+1+ps}{ps}\rfloor}{}_{n+1-(i-1)ps-s(v-1)-h}C_{m}. \tag{2.9}$$

$$\sum_{i=1}^{\lfloor\frac{n-m-s(v-1)-h+ps}{ps}\rfloor}{}_{n-(i-1)ps-s(v-1)-h}C_{m}. \tag{2.10}$$

$$\sum_{i=1}^{\lfloor\frac{n-m-s(v-1)-h+1+ps}{ps}\rfloor}{}_{n-(i-1)ps-s(v-1)-h}C_{m-1}. \tag{2.11}$$

For i such that $\lfloor \frac{n-m-s(v-1)-h+1+ps}{ps} \rfloor \geq \lfloor \frac{n-m-s(v-1)-h+ps}{ps} \rfloor \geq i \geq 1$, by using Lemma 2 we get the following equation.

$$_{n+1-(i-1)ps-s(v-1)-h}C_m =_{n-(i-1)ps-s(v-1)-h} C_m +_{n-(i-1)ps-s(v-1)-h} C_{m-1}. \tag{2.12}$$

We have two cases.
Case (a) If $\lfloor \frac{n-m-s(v-1)-h+1+ps}{ps} \rfloor = \lfloor \frac{n-m-s(v-1)-h+ps}{ps} \rfloor$, then by (2.12) the proof is finished.
Case (b) We suppose that

$$\lfloor \frac{n-m-s(v-1)-h+1+ps}{ps} \rfloor > \lfloor \frac{n-m-s(v-1)-h+ps}{ps} \rfloor. \tag{2.13}$$

By (2.13) we know that (2.10) does not have the $\lfloor \frac{n-m-s(v-1)-h+1+ps}{ps} \rfloor$th term, and hence we compare the $\lfloor \frac{n-m-s(v-1)-h+1+ps}{ps} \rfloor$th term of (2.9) and (2.11).

By (2.13)

$$\lfloor \frac{n-m-s(v-1)-h+1+ps}{ps} \rfloor = \frac{n-m-s(v-1)-h+1+ps}{ps}. \tag{2.14}$$

Let $i = \lfloor \frac{n-m-s(v-1)-h+1+ps}{ps} \rfloor$. Then by (2.14)

$$\begin{aligned}
&_{n+1-(i-1)ps-s(v-1)-h}C_m \\
&=_{n+1-(n-m-s(v-1)-h+1)-s(v-1)-h} C_m \\
&=_m C_m = 1 =_{m-1} C_{m-1} \\
&=_{n-(n-m-s(v-1)-h+1)-s(v-1)-h} C_{m-1} \\
&=_{n-(i-1)ps-s(v-1)-h} C_{m-1},
\end{aligned}$$

and hence the i th term of (2.9) is equal to the i th term of (2.11). Therefore, (2.9) is equal to the sum of (2.10) and (2.11), and the proof of this lemma is finished. □

Theorem 4. $f(p,n,m,s,v) = \frac{U(p,n,m,s,v)}{_nC_m}$.

Proof. This is direct from Definition 5. □

By Lemma 4 and Theorem 4, $\{f(p,n,m,s,v) : m \leq n, n = 1,2,...\}$ has a pattern similar to Pascal's triangle for fixed positive integers p,s,v.

3. Fibonacci-like Sequence Produced by Pascal-like Sequence

We generalize the result on Fibonacci-like sequence presented in Section 1. In the remainder of this article, we suppose that $v = 1$ to make the argument simpler.

Lemma 5. Let t be a non-negative integer. Then, we have the following (i) and (ii).
(i) $U(p, tps+u, 1, s, 1) = ts+u$ for any natural number u such that $0 \leq u \leq s$.
(ii) $U(p, tps+u, 1, s, 1) = ts+s$ for any natural number u such that $s < u \leq ps$.

Proof. For a natural number u such that $0 \leq u \leq ps$, by Theorem 3

$$U(p, tps+u, 1, s, 1) = \sum_{h=1}^{s} \sum_{i=1}^{\lfloor \frac{tps+u-h+ps}{ps} \rfloor} {}_{tps+u-(i-1)ps-h}C_0 = \sum_{h=1}^{s} \lfloor \frac{tps+u-h+ps}{ps} \rfloor. \tag{3.1}$$

(i) If $1 \leq u \leq s$, then (3.1) is $\sum_{h=1}^{u}(t+1) + \sum_{h=u+1}^{s} t = st+u$.
(ii) If $s < u \leq ps$, then (3.1) is $\sum_{h=1}^{s}(t+1) = st+s$. □

Lemma 6. $U(p,u,u,s,1) = 1$ for any natural number u.

Proof. By Theorem 3, $U(p,u,u,s,1)=\sum_{h=1}^{s}\sum_{i=1}^{\lfloor\frac{-h+1+ps}{ps}\rfloor}{}_{u-(i-1)ps-h}C_{u-1}$
$=\sum_{h=1}^{1}\sum_{i=1}^{\lfloor\frac{ps}{ps}\rfloor}{}_{u-(i-1)ps-1}C_{u-1}={}_{u-1}C_{u-1}=1.$ □

We generalize the sequence introduced in Example 1, and define $B_{p,s}(n), n=1,2,3,...$ in Definition 6.

Definition 6. For natural numbers p and s, let

$$B_{p,s}(n)=\sum_{k=0}^{\lfloor\frac{n-1}{2}\rfloor}U(p,n-k,k+1,s,1). \tag{3.2}$$

(3.2) is a generalization of (1.1).

Theorem 5. For a natural number n such that $n\geq 3$,

$$B_{p,s}(n)=B_{p,s}(n-1)+B_{p,s}(n-2)+\begin{cases}1 & (1\leq n\leq s \pmod{ps})\\ 0 & (n=0 \text{ or } n\geq s+1 \pmod{ps})\,.\end{cases} \tag{3.3}$$

Proof. Let $n=tps+h$ for non-negative integers t,h such that $0\leq h<ps$.

$$B_{p,s}(tps+h)=U(p,tps+h,1,s,1)+U(p,tps+h-1,2,s,1)+...,$$
$$+U(p,tps+h-t_1,t_1+1,1), \tag{3.4}$$
$$B_{p,s}(tps+h-1)=U(p,tps+h-1,1,s,1)+U(p,tps+h-2,2,s,1)+...,$$
$$+U(p,tps+h-1-t_2,t_2+1,1) \tag{3.5}$$
and
$$B_{p,s}(tps+h-2)=U(p,tps+h-2,1,s,1)+U(p,tps+h-3,2,s,1)+...,$$
$$+U(p,tps+h-2-t_3,t_3+1,1), \tag{3.6}$$
$$\text{where } t_1=\lfloor\frac{tps+h-1}{2}\rfloor, t_2=\lfloor\frac{tps+h-2}{2}\rfloor \text{ and } t_3=\lfloor\frac{tps+h-3}{2}\rfloor. \tag{3.7}$$

Case (i) First, we assume that $tps+h$ is an even number, then $t_1=t_2=t_3+1$. By Lemma 4, we have for $k=1,2,...,t_1$
$U(p,tps+h-k,k+1,s,1)=U(p,tps+h-1-k,k+1,s,1)+U(p,tps+h-1-k,k,s,1)$,
and hence the $k+1$ th term of (3.4) is equal to the sum of the $k+1$ th term of (3.5) and the k th term of (3.6). Therefore, the sum of the 2nd, 3rd, ..., t_1+1th term of (3.4) is equal to the sum of the 2nd, 3rd, ..., t_1+1th term of (3.5) and the 1st, 2nd, ..., t_1th term of (3.6). Therefore we only have to compare the 1st term of (3.4) and (3.5). The 1st term of (3.4) is $U(p,tps+h,1,s,1)$, and the 1st term of (3.5) is $U(p,tps+h-1,1,s,1)$. Therefore,

$$B_{p,s}(n)-(B_{p,s}(n-1)+B_{p,s}(n-2))=U(p,tps+h,1,s,1)-U(p,tps+h-1,1,s,1). \tag{3.8}$$

We have two subcases.
Subcase (a) If $1\leq n\leq s$ (mod ps), then we have $1\leq h\leq s$ and $0\leq h-1\leq s-1$. Then, by (i) of Lemma 5 (3.8) is equal to $ts+h$-$(ts+h-1)=1$.
Subcase (b) If $n=0$ or $n\geq s+1$ (mod ps), then $h=0$ or $s+1\leq h$. We have three subsubcases.
Subsubcase $(b.1)$ Suppose that $h=0$. Then, by $(i),(ii)$ of Lemma 5 (3.8) is equal to $U(p,tps,1,s,1)-U(p,(t-1)ps+ps-1,1,s,1)=ts-((t-1)s+s)=0$.
Subsubcase $(b.2)$ Suppose that $h=s+1$. Then by $(ii),(i)$ of Lemma 5 (3.8) is equal to $U(p,tps+s+1,1,s,1)-U(p,tps+s,1,s,1)=ts+s-(ts+s)=0$.
Subsubcase $(b.3)$ Suppose that $h>s+1$. Then by (ii) of Lemma 5 (3.8) is equal to $U(p,tps+h,1,s,1)-U(p,tps+h-1,1,s,1)=ts+s-(ts+s)=0$.
Therefore we prove this Theorem.
Case (ii) Next we suppose that $tps+h$ is an odd number. Then $t_1=t_2+1=t_3+1$.
By Lemma 4 we have for $k=1,2,...,t_2$
$U(p,tps+h-k,k+1,s,1)=U(p,tps+h-1-k,k+1,s,1)+U(p,tps+h-1-k,k,s,1)$, and hence the $k+1$ th term of (3.4) is equal to the sum of the $k+1$ th term of (3.5) and the k th term of (3.6). Then the sum of the 2nd, 3rd, ..., t_2+1th term of (3.4) is equal to the sum of the 2nd, 3rd, ..., t_2+1th term of (3.5)

and the 1st, 2nd, ..., t_3th term of (3.6).
Therefore we only have to compare the 1st and the $t_1 + 1 = t_2 + 2$ th terms of (3.4), the 1st term of (3.5) and the $t_3 + 1$ th term of (3.6). Since $tps + h$ is an odd number, $t_1 = \lfloor \frac{tps+h-1}{2} \rfloor = \frac{tps+h-1}{2}$ and $t_3 = \lfloor \frac{tps+h-3}{2} \rfloor = \frac{tps+h-3}{2}$. Therefore $tps+h-t_1 = t_1+1$ and $tps+h-2-t_3 = t_3+1$, and hence by Lemma 6

$$U(p, tps+h-t_1, t_1+1, s, 1) = U(p, tps+h-2-t_3, t_3+1, s, 1) = 1. \tag{3.9}$$

Therefore by (3.9) and Lemma 5

$$\begin{aligned}
&B_{p,s}(n) - (B_{p,s}(n-1) + B_{p,s}(n-2)) \\
&= (U(p, tps+h, 1, s, 1) + U(p, tps+h-t_1, t_1+1, s, 1)) \\
&- (U(p, tps+h-1, 1, s, 1) + U(p, tps+h-2-t_3, t_3+1, s, 1)) \\
&= U(p, tps+h, 1, s, 1) - U(p, tps+h-1, 1, s, 1).
\end{aligned} \tag{3.10}$$

Then we prove this theorem using a method that is the similar to the method used in Case (i). □

4. The Properties of the Sequence $B_{p,s}(n)$

There are some interesting properties of $B_{p,1}(n)$.

Lemma 7. Let p be a natural number. Then, we have the following (i), (ii) and (iii).

$(i) B_{p,1}(1) = B_{p,1}(2) = 1.$
$(ii) B_{p,2}(1) = 1.$
$(iii) B_{p,2}(2) = 2.$

Proof. Since $\lfloor \frac{n-1}{2} \rfloor = 0$ for $n = 1, 2$, by Definition 6 for any natural number p

$$B_{p,s}(1) = U(p, 1, 1, s, 1) \tag{4.1}$$

and

$$B_{p,s}(2) = U(p, 2, 1, s, 1). \tag{4.2}$$

By (i) of Lemma 5, we have for $s = 1, 2$

$$U(p, 1, 1, s, 1) = 1. \tag{4.3}$$

By (ii) of Lemma 5, we have for $s = 1$

$$U(p, 2, 1, s, 1) = 1. \tag{4.4}$$

By (i) of Lemma 5, we have for $s = 2$

$$U(p, 2, 1, s, 1) = 2. \tag{4.5}$$

By (4.1), (4.2), (4.3) and (4.4) we have (i), and by (4.1) and (4.3) we have (ii). By (4.2) and (4.5) we have (iii). □

Example 3. Here we present Fibonacci-like sequence $B_{p,s}(n)$ for $s = 1, 2$ and Fibonacci sequence $F(n)$.
(1) $F(n)$ is {1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610}.
By Theorem 5 and Lemma 7, for $s = 1$ we have $B_{p,1}(1) = B_{p,1}(2) = 1$, and for a natural number n such that $n \geq 3$,

$$B_{p,1}(n) = B_{p,1}(n-1) + B_{p,1}(n-2) + \begin{cases} 1 & (n = 1 \pmod{p}) \\ 0 & (n = 0 \text{ or } n \geq 2 \pmod{p}). \end{cases} \tag{4.6}$$

Then,
(2) $B_{2,1}(n)$ is {1, 1, 3, 4, 8, 12, 21, 33, 55, 88, 144, 232, 377, 609, 987}.

(3) $B_{3,1}(n)$ is {1, 1, 2, 4, 6, 10, 17, 27, 44, 72, 116, 188, 305, 493, 798}.
(4) $B_{4,1}(n)$ is {1, 1, 2, 3, 6, 9, 15, 24, 40, 64, 104, 168, 273, 441, 714}.
There are well known relations between $B_{2,1}(n)$, $B_{3,1}(n)$ and the Fibonacci sequence $F(n)$.
$B_{2,1}(n) = F(n+1) - \frac{(1-(-1)^{n+1})}{2}$.
$B_{3,1}(n) = \lfloor(\frac{F(n+2)}{2})\rfloor$.
By Theorem 5 and Lemma 7, for $s = 2$ we have $B_{p,2}(1) = 1, B_{p,2}(2) = 2$, and for a natural number n such that $n \geq 3$,

$$B_{p,2}(n) = B_{p,2}(n-1) + B_{p,2}(n-2) + \begin{cases} 1 & (1 \leq n \leq 2 \pmod{2p}) \\ 0 & (n = 0 \text{ or } n \geq 3 \pmod{2p}) \,. \end{cases} \tag{4.7}$$

Then,
(5) $B_{2,2}(n)$ is {1, 2, 3, 5, 9, 15, 24, 39, 64, 104, 168, 272, 441, 714, 1155 }.
(6) $B_{3,2}(n)$ is {1, 2, 3, 5, 8, 13, 22, 36, 58, 94, 152, 246, 399, 646, 1045 }.
(7) $B_{4,2}(n)$ is {1, 2, 3, 5, 8, 13, 21, 34, 56, 91, 147, 238, 385, 623, 1008 }. $B_{2,2}(n)$ and $B_{3,2}(n)$ have simple formulas.
$B_{2,2}(n) = \lfloor((1+\sqrt{5})/2)^{n+3})/5\rfloor$
$B_{3,2}(n)$ is $\lfloor\frac{F(n+4)}{4}\rfloor$.

5. The General Case of Two Persons

In previous sections, the first player to draw out a red card loses the game, and the game ends at this round. In the remainder of this article, we change this rule. To make the argument more simple, we study a game between two players Θ_1 and Θ_2. For the same reason, we calculate only the combinations for Player Θ_1.

Definition 7. Two players Θ_1 and Θ_2 take turns, and draw a card s_1 and s_2 times respectively. Each player stores the cards that he or she draws. If the number of the stored cards for Θ_1 or Θ_2 reaches g_1 or g_2, then Θ_1 or Θ_2 loses the game respectively.

We denote by $U(n, m, s_1, s_2, g_1, g_2)$ the possible arrangements of positions of the m red cards that would lead to Player Θ_1 losing the game.

If $m < g_1$, then Player Θ_1 never lose in the game. Therefore, we assume that $m \geq g_1$ when we calculate $U(n, m, s_1, s_2, g_1, g_2)$.

Theorem 6. We assume that

$$m \geq g_1. \tag{5.1}$$

Then,

$$U(n, m, s_1, s_2, g_1, g_2) = \sum_{h=1}^{s_1} \Bigg(\sum_{k=\lceil \frac{g_1-h}{s_1} \rceil+1}^{\min(\lfloor \frac{n-m+g_1-h}{s_1} \rfloor+1, \lfloor \frac{n-h}{s_1+s_2} \rfloor+1, \lfloor \frac{n-m+g_1+g_2-h-1}{s_1+s_2} \rfloor+1)} \sum_{v=\max(0, m-n+(k-1)(s_1+s_2)-g_1+h)}^{\min(g_2-1, m-g_1, (k-1)s_2)} {}_{(k-1)s_2}C_v \times {}_{h+(k-1)s_1-1}C_{g1-1} \times {}_{n-(k-1)(s_1+s_2)-h}C_{m-v-g_1} \Bigg) \tag{5.2}$$

Proof. Suppose that player Θ_1 loses the game by collecting g_1 cards in the $(k-1)(s_1+s_2)+h$ th round for natural numbers k and h such that $1 \leq h \leq s_1$. Since the number of rounds is smaller or equal to the number of cards,

$$(k-1)(s_1+s_2)+h \leq n.$$

Therefore,

$$k \leq \lfloor \frac{n-h}{s_1+s_2} \rfloor + 1. \tag{5.3}$$

Player Θ_1 collects $g_1 - 1$ red cards before player Θ_1 draws a red card in the $(k-1)(s_1+s_2)+h$ th round to lose the game. The number of rounds player Θ_1 plays before that is $(k-1)s_1 + h - 1$. Hence,

$$(k-1)s_1 + h - 1 \geq g_1 - 1.$$

Therefore

$$k \geq \lceil \frac{g_1 - h}{s_1} \rceil + 1. \tag{5.4}$$

The number of combination of $g_1 - 1$ red cards in these $(k-1)s_1 + h - 1$ rounds is

$${}_{(k-1)s_1+h-1}C_{g_1-1}.$$

Suppose that player Θ_2 collects v red cards before player Θ_1 collects g_1 red cards. The number of rounds player Θ_2 plays before that is $(k-1)s_2$, and hence the number of combination of is

$${}_{(k-1)s_2}C_v.$$

Clearly,

$$(k-1)s_2 \geq v. \tag{5.5}$$

A non-negative integer v should be less than g_2. Otherwise Θ_2 loses the game. Therefore

$$g_2 - 1 \geq v. \tag{5.6}$$

Since player Θ_2 draws v red cards from remaining $m - g_1$ red cards,

$$m - g_1 \geq v. \tag{5.7}$$

The other $m - v - g_1$ red cards are positioned after the $(k-1)(s_1+s_2)+h$ th place, so

$$n - ((k-1)(s_1+s_2)+h) \geq m - v - g_1$$

, and there are

$${}_{n-((k-1)(s_1+s_2)+h)}C_{m-v-g_1}$$

ways to arranging the cards in this way. Then we have

$$v \geq m - n + (k-1)(s_1+s_2) - g_1 + h. \tag{5.8}$$

Since $v \geq 0$, by (5.8)

$$v \geq \max(0, m - n + (k-1)(s_1+s_2) - g_1 + h). \tag{5.9}$$

By (5.5), (5.6), (5.7) and (5.9),

$$(k-1)s_2 \geq m - n - g_1 + h + (k-1)(s_1+s_2), \tag{5.10}$$
$$g_2 - 1 \geq m - n - g_1 + h + (k-1)(s_1+s_2) \tag{5.11}$$

and

$$m - g_1 \geq m - n - g_1 + h + (k-1)(s_1+s_2). \tag{5.12}$$

Note that by the definitions of s_2, g_2 and (5.1) we have $(k-1)s_2 \geq 0$, $g_2 - 1 \geq 0$ and $m - g_1 \geq 0$. By (5.10), (5.11) and (5.12) we have

$$\frac{g_1 - h - m + n}{s_1} + 1 = \frac{g_1 - h - m + n + s_1}{s_1} \geq k, \tag{5.13}$$
$$\frac{g_1 + g_2 - h - m + n - 1}{s_1 + s_2} + 1 = \frac{g_1 + g_2 - h - m + n + s_1 + s_2 - 1}{s_1 + s_2} \geq k \tag{5.14}$$

and

$$\frac{n - h}{s_1 + s_2} + 1 \geq k. \tag{5.15}$$

(5.15) is the same as (5.3). By (5.3),(5.4),(5.5),(5.6),(5.7),(5.9),(5.13) and (5.14) we have (5.2).

□

Definition 8. We fix natural numbers s_1, s_2, g_1, g_2 and h such that $1 \leq h \leq s_1$, and for natural numbers n, m, k we define

$$LOW1 = \lceil \frac{g_1 - h}{s_1} \rceil + 1, \tag{5.16}$$

$$UPP1(n,m) = \min(\lfloor \frac{n - m + g_1 - h}{s_1} \rfloor + 1, \lfloor \frac{n - h}{s_1 + s_2} \rfloor + 1, \lfloor \frac{n - m + g_1 + g_2 - h - 1}{s_1 + s_2} \rfloor + 1), \tag{5.17}$$

$$LOW2(n,m,k) = \max(0, m - n + (k-1)(s_1 + s_2) - g_1 + h) \tag{5.18}$$

and

$$UPP2(n,m,k) = \min(g_2 - 1, m - g_1, (k-1)s_2). \tag{5.19}$$

Lemma 8. Suppose that natural numbers a, b, a', b' satisfy the following conditions.

$$\min(a,b) > \min(a',b'), \tag{5.20}$$

$$a' + 1 \geq a \geq a' \tag{5.21}$$

and

$$b' + 1 \geq b \geq b'. \tag{5.22}$$

Then, we have (i) or (ii).
(i) $b \geq a > a'$.
(ii) $a \geq b > b'$.

Proof. By (5.21) and (5.22) we have the following four cases.
$Case(a)$ Suppose that $a = a'$ and $b = b'$. Then this contradicts (5.20).
$Case(b)$ Suppose that $a = a' + 1$ and $b = b' + 1$. If $b \geq a$, then we have $b \geq a > a'$. This is (i). If $a \geq b$, then we have $a \geq b > b'$. This is (ii).
$Case(c)$ Suppose that $a = a' + 1$ and $b = b'$. Then, by (5.20) $b \geq a > a'$. This is (i).
$Case(d)$ Suppose that $a = a'$ and $b = b' + 1$. Then, by (5.20) $a \geq b > b'$. This is (ii). □

Example 4. Here, we present the Pascal-like triangles formed from $\{U(n, m, s_1, s_2, g_1, g_2), 1 \leq m \leq n, n = 1, 2, ..., 9, 10\}$, where $s_1 = 2, s_2 = 4, g_1 = 2, g_2 = 3$, in Figure 5 and Figure 6.

0
0 1
0 1 1
0 1 2 1
0 1 3 3 1
0 1 4 6 4 1
0 3 13 22 10 5 1
0 6 28 53 32 15 6 1
0 6 34 81 85 47 21 7 1
0 6 40 115 166 132 68 28 8 1

Figure 5.

0
0 1
0 1 1
0 1 2 1
0 1 3 3 1
0 1 4 6 4 1
0 3 13 22 10 5 1
0 6 28 53 32 15 6 1
0 6 34 81 85 47 21 7 1
0 6 40 115 166 132 68 28 8 1

Figure 6.

It is clear that some parts of these triangles satisfy (5.23) and other parts do not.

$$U(n, m, s_1, s_2, g_1, g_2) = U(n-1, m-1, s_1, s_2, g_1, g_2) + U(n-1, m, s_1, s_2, g_1, g_2). \tag{5.23}$$

By Figure 5, it seems that we have Pascal-like properties of (5.23) for a natural number m that satisfies (5.24).

$$m \geq g_1 + g_2. \tag{5.24}$$

We prove this fact in Theorem 7.

By Figure 6, it seems that we have Pascal-like properties of (5.23) for a natural number n such that

$$n = (s_1+s_2)(t-1)+s_1+u \tag{5.25}$$

for natural numbers t, u such that $1 \le u \le s_2$. We prove this fact in Theorem 8. For $s_1 = 2, s_2 = 4, g_1 = 2, g_2 = 3$ by (5.25) we have (5.23) for $n = 3, 4, 5, 6, 9, 10, \dots$.

Theorem 7. Suppose that

$$m \ge g_1 + g_2. \tag{5.26}$$

Then

$$U(n,m,s_1,s_2,g_1,g_2) = U(n-1,m-1,s_1,s_2,g_1,g_2) + U(n-1,m,s_1,s_2,g_1,g_2). \tag{5.27}$$

Proof. By (5.16), (5.18), (5.17), (5.19) and Theorem 6

$$\begin{aligned} &U(n,m,s_1,s_2,g_1,g_2) \\ &= \sum_{h=1}^{s_1} \sum_{k=LOW1}^{UPP1(n,m)} \sum_{v=LOW2(n,m,k)}^{UPP2(n,m,k)} {}_{(k-1)s_2}C_v \times {}_{h+(k-1)s_1-1}C_{g_1-1} \times {}_{n-(k-1)(s_1+s_2)-h}C_{m-v-g_1}, \end{aligned} \tag{5.28}$$

$$\begin{aligned} &U(n-1,m-1,s_1,s_2,g_1,g_2) \\ &= \sum_{h=1}^{s_1} \sum_{k=LOW1}^{UPP1(n-1,m-1)} \sum_{v=LOW2(n-1,m-1,k)}^{UPP2(n-1,m-1,k)} {}_{(k-1)s_2}C_v \times {}_{h+(k-1)s_1-1}C_{g_1-1} \times {}_{n-1-(k-1)(s_1+s_2)-h}C_{m-1-v-g_1} \end{aligned} \tag{5.29}$$

and

$$\begin{aligned} &U(n-1,m,s_1,s_2,g_1,g_2) \\ &= \sum_{h=1}^{s_1} \sum_{k=LOW1}^{UPP1(n-1,m)} \sum_{v=LOW2(n-1,m,k)}^{UPP2(n-1,m,k)} {}_{(k-1)s_2}C_v \times {}_{h+(k-1)s_1-1}C_{g_1-1} \times {}_{n-1-(k-1)(s_1+s_2)-h}C_{m-v-g_1}. \end{aligned} \tag{5.30}$$

We are to prove that (5.28) is equal to the sum of (5.29) and (5.30). Let h be a natural number such that $1 \le h \le s_1$, and we fix h. By (5.26) we have $n-1-h \ge n-m+g_1+g_2-h-1$, and hence by (5.17) we have

$$UPP1(n,m) = UPP1(n-1,m-1) = UPP1(n-1,m) \tag{5.31}$$

or

$$UPP1(n,m) = UPP1(n-1,m-1) > UPP1(n-1,m). \tag{5.32}$$

We have two cases.
Case(a) We suppose that (5.31) is valid. Let k be a natural number such that $LOW1 \le k \le UPP1(n,m)$. If we prove that (5.33) is equal to the sum of (5.34) and (5.35), then we have (5.27).

$$\sum_{v=LOW2(n,m,k)}^{UPP2(n,m,k)} {}_{(k-1)s_2}C_v \times {}_{h+(k-1)s_1-1}C_{g_1-1} \times {}_{n-(k-1)(s_1+s_2)-h}C_{m-v-g_1}, \tag{5.33}$$

$$\sum_{v=LOW2(n-1,m-1,k)}^{UPP2(n-1,m-1,k)} {}_{(k-1)s_2}C_v \times {}_{h+(k-1)s_1-1}C_{g_1-1} \times {}_{n-1-(k-1)(s_1+s_2)-h}C_{m-1-v-g_1} \tag{5.34}$$

and

$$\sum_{v=LOW2(n-1,m,k)}^{UPP2(n-1,m,k)} {}_{(k-1)s_2}C_v \times {}_{h+(k-1)s_1-1}C_{g_1-1} \times {}_{n-1-(k-1)(s_1+s_2)-h}C_{m-v-g_1}. \tag{5.35}$$

By (5.26) we have $m-g_1 \geq g_2-1$ and $(m-1)-g_1 \geq g_2-1$. Therefore, by (5.19)

$$UPP2(n,m,k) = UPP2(n-1,m-1,k) = UPP2(n-1,m,k). \tag{5.36}$$

By (5.18)

$$LOW2(n,m,k) = LOW2(n-1,m-1,k) \leq LOW2(n-1,m,k) \tag{5.37}$$

, and hence we have the following subcases.
$Subcase(a.1)$ Suppose that $LOW2(n,m,k) = LOW2(n-1,m-1,k) = LOW2(n-1,m,k)$.

For v such that $LOW2(n,m,k) \leq v \leq UPP2(n,m,k)$, by Lemma 2 we have

$$\begin{aligned} &{}_{(k-1)s_2}C_v \times {}_{h+(k-1)s_1-1}C_{g1-1} \times {}_{n-(k-1)(s_1+s_2)-h}C_{m-v-g_1} \\ &= {}_{(k-1)s_2}C_v \times {}_{h+(k-1)s_1-1}C_{g1-1} \times {}_{n-1-(k-1)(s_1+s_2)-h}C_{m-1-v-g_1} \\ &+ {}_{(k-1)s_2}C_v \times {}_{h+(k-1)s_1-1}C_{g1-1} \times {}_{n-1-(k-1)(s_1+s_2)-h}C_{m-v-g_1}. \end{aligned} \tag{5.38}$$

Therefore we have (5.33) = (5.34) + (5.35), and we have (5.27).
$Subcase(a.2)$ Suppose that $LOW2(n,m,k) = LOW2(n-1,m-1,k) < LOW2(n-1,m,k)$. For v such that $LOW2(n-1,m,k) \leq v \leq UPP2(n,m,k)$, by Lemma 2 we have (5.38).
Therefore the sum of the second, the third, ..., the last term of (5.33) is equal to the sum of the second, the third,..., the last term of (5.34) and the first, the second, ...,the last term of (5.35), and we have to prove that the first term of (5.33) is equal to the the first term of (5.34) to prove that (5.33) is the sum of (5.34) and (5.35). Suppose that $0 > m-n+(k-1)(s_1+s_2)-g_1+h$. Then, $0 \geq m-n+1+(k-1)(s_1+s_2)-g_1+h$, and we have $LOW2(n,m,k) = LOW2(n-1,m,k) = 0$. This contradicts $LOW2(n,m,k) = LOW2(n-1,m-1,k) < LOW2(n-1,m,k)$. Therefore $0 \leq m-n+(k-1)(s_1+s_2)-g_1+h$. Let $v = LOW2(n,m,k) = m-n+(k-1)(s_1+s_2)-g_1+h$, then $n-(k-1)(s_1+s_2)-h = m-v-g_1$. Therefore we have ${}_{n-(k-1)(s_1+s_2)-h}C_{m-v-g_1} = 1 = {}_{n-1-(k-1)(s_1+s_2)-h}C_{m-1-v-g_1}$, and we have

$$\begin{aligned} &{}_{(k-1)s_2}C_v \times {}_{h+(k-1)s_1-1}C_{g1-1} \times {}_{n-(k-1)(s_1+s_2)-h}C_{m-v-g_1} \\ &= {}_{(k-1)s_2}C_v \times {}_{h+(k-1)s_1-1}C_{g1-1} \times {}_{n-1-(k-1)(s_1+s_2)-h}C_{m-1-v-g_1}. \end{aligned}$$

This shows that the first term of (5.33) is equal to the the first term of (5.34).
$Case(b)$ We suppose that (5.32) is valid. Let $LOW1 \leq k \leq UPP1(n-1,m)$, then by the method that is similar to the one used in $(a.1)$ and $(a.2)$ we have (5.33) = (5.34) + (5.35). Next, we prove (5.33) = (5.34) for

$$k = UPP1(n,m) = UPP1(n-1,m-1) > UPP1(n-1,m). \tag{5.39}$$

By (5.17) and (5.26), $\lfloor \frac{n-h}{s_1+s_2} \rfloor + 1 \geq \lfloor \frac{n-m+g_1+g_2-h-1}{s_1+s_2} \rfloor + 1$ and $\lfloor \frac{n-1-h}{s_1+s_2} \rfloor + 1 \geq \lfloor \frac{n-1-m+g_1+g_2-h-1}{s_1+s_2} \rfloor + 1$, and hence by (5.39) we have

$$\begin{aligned} &k = \min(\lfloor \frac{n-m+g_1-h}{s_1} \rfloor + 1, \lfloor \frac{n-m+g_1+g_2-h-1}{s_1+s_2} \rfloor + 1) \\ &> \min(\lfloor \frac{n-1-m+g_1-h}{s_1} \rfloor + 1, \lfloor \frac{n-1-m+g_1+g_2-h-1}{s_1+s_2} \rfloor + 1). \end{aligned}$$

By Lemma 8 we have the following two subcases.
$Subcase(b.1)$ We suppose that

$$\begin{aligned} &\lfloor \frac{n-m+g_1-h}{s_1} \rfloor + 1 \\ &\geq \lfloor \frac{n-m+g_1+g_2-h-1}{s_1+s_2} \rfloor + 1 = k > \lfloor \frac{n-1-m+g_1+g_2-h-1}{s_1+s_2} \rfloor + 1. \end{aligned} \tag{5.40}$$

By (5.40) we have

$$n-m+g_1+g_2-h-1 = (k-1)(s_1+s_2), \tag{5.41}$$

and hence

$$m-n+(k-1)(s_1+s_2)-g_1+h = g_2-1. \tag{5.42}$$

By (5.32) we have to prove that (5.33) = (5.34) for k that satisfies (5.39). By (5.36) and (5.37), we choose a non-negative integer v such that $LOW2(n-1,m-1,k) = LOW2(n,m,k)$ $\leq v \leq UPP2(n,m,k) = UPP2(n-1,m-1,k)$. Then by (5.18), (5.19) and (5.42) $g_2 - 1 = m - n + (k-1)(s_1+s_2) - g_1 + h \leq LOW2(n,m,k) \leq v \leq UPP2(n,m,k) \leq g_2 - 1$, and hence $v = g_2 - 1$. Then (5.33) and (5.34) consist of a single term. Therefore by (5.41) $n - (k-1)(s_1+s_2) - h = m - g_1 - g_2 + 1 = m - (g_2 - 1) - g1 = m - v - g_1$, and we have ${}_{n-(k-1)(s_1+s_2)-h}C_{m-v-g_1} = 1 = {}_{n-1-(k-1)(s_1+s_2)-h}C_{m-1-v-g_1}$.

Therefore we have (5.33) = (5.34).

Subcase$(b.2)$ We suppose that

$$\lfloor\frac{n-m+g_1+g_2-h-1}{s_1+s_2}\rfloor + 1$$
$$\geq \lfloor\frac{n-m+g_1-h}{s_1}\rfloor + 1 = k > \lfloor\frac{n-1-m+g_1-h}{s_1}\rfloor + 1. \tag{5.43}$$

By (5.43) we have

$$n - m + g_1 - h = (k-1)s_1. \tag{5.44}$$

By (5.44) we have

$$m - n + (k-1)(s_1+s_2) - g_1 + h = (k-1)s_2. \tag{5.45}$$

By (5.36) and (5.37) we choose a non-negative integer v such that $LOW2(n-1,m-1,k) = LOW2(n,m,k) \leq v \leq UPP2(n,m,k) = UPP2(n-1,m-1,k)$. Then by (5.18), (5.19) and (5.45), $(k-1)s_2 = m - n + (k-1)(s_1+s_2) - g_1 + h \leq LOW2(n,m,k) \leq v \leq UPP2(n,m,k) \leq (k-1)s_2$, and hence $v = (k-1)s_2$. Then (5.33) and (5.34) consist of a single term. Therefore by (5.45) $n-(k-1)(s_1+s_2)-h = m-g_1-(k-1)s_2 = m-v-g_1$, and we have

$${}_{n-(k-1)(s_1+s_2)-h}C_{m-v-g_1} = 1 = {}_{n-1-(k-1)(s_1+s_2)-h}C_{m-1-v-g_1}.$$

Therefore we have (5.33) = (5.34). □

Theorem 8. Suppose that

$$n = (s_1+s_2)(t-1) + s_1 + u \tag{5.46}$$

for natural numbers t, u such that $1 \leq u \leq s_2$.

Then

$$U(n,m,s_1,s_2,g_1,g_2) = U(n-1,m-1,s_1,s_2,g_1,g_2) + U(n-1,m,s_1,s_2,g_1,g_2). \tag{5.47}$$

Proof. If $m \geq g_1 + g_2$, then by Theorem 7 we have (5.47). Therefore, in this proof we assume that

$$m < g_1 + g_2. \tag{5.48}$$

By (5.48) $n - h \leq n - m + g_1 + g_2 - h - 1$, and hence by Definition 8

$$UPP1(n,m) = \min(\lfloor\frac{n-m+g_1-h}{s_1}\rfloor + 1, \lfloor\frac{n-h}{s_1+s_2}\rfloor + 1), \tag{5.49}$$

$$UPP1(n-1,m-1) = \min(\lfloor\frac{n-m+g_1-h}{s_1}\rfloor + 1, \lfloor\frac{n-1-h}{s_1+s_2}\rfloor + 1) \tag{5.50}$$

and

$$UPP1(n-1,m) = \min(\lfloor\frac{n-1-m+g_1-h}{s_1}\rfloor + 1, \lfloor\frac{n-1-h}{s_1+s_2}\rfloor + 1). \tag{5.51}$$

Since $1 \leq h \leq s_1$ and $1 \leq u \leq s_2$, by (5.46) we have $(s_1+s_2)(t-1) \leq n-h-1 < n-h \leq (s_1+s_2)(t-1) + s_1 + s_2 - 1$. Therefore

$$\lfloor\frac{n-h}{s_1+s_2}\rfloor + 1 = \lfloor\frac{n-1-h}{s_1+s_2}\rfloor + 1. \tag{5.52}$$

By (5.49), (5.50), (5.51) and (5.52) we have

$$UPP1(n,m) = UPP1(n-1,m-1) \geq UPP1(n-1,m). \tag{5.53}$$

(a) Suppose that $UPP1(n,m) = UPP1(n-1,m-1) = UPP1(n-1,m)$. We fix k such that $LOW1 \leq k \leq UPP1(n,m)$.
We are to prove that (5.54) is the sum of (5.55) and (5.56).

$$\sum_{v=LOW2(n,m,k)}^{UPP2(n,m,k)} ({}_{(k-1)s_2}C_v \times {}_{h+(k-1)s_1-1}C_{g1-1} \times {}_{n-(k-1)(s_1+s_2)-h}C_{m-v-g_1}), \tag{5.54}$$

$$\sum_{v=LOW2(n-1,m-1,k)}^{UPP2(n-1,m-1,k)} ({}_{(k-1)s_2}C_v \times {}_{h+(k-1)s_1-1}C_{g1-1} \times {}_{n-1-(k-1)(s_1+s_2)-h}C_{m-1-v-g_1}) \tag{5.55}$$

and

$$\sum_{v=LOW2(n-1,m,k)}^{UPP2(n-1,m,k)} ({}_{(k-1)s_2}C_v \times {}_{h+(k-1)s_1-1}C_{g1-1} \times {}_{n-1-(k-1)(s_1+s_2)-h}C_{m-v-g_1}). \tag{5.56}$$

For a fixed v, we compare (5.57), (5.58) and (5.59).

$${}_{(k-1)s_2}C_v \times {}_{h+(k-1)s_1-1}C_{g1-1} \times {}_{n-(k-1)(s_1+s_2)-h}C_{m-v-g_1}, \tag{5.57}$$

$${}_{(k-1)s_2}C_v \times {}_{h+(k-1)s_1-1}C_{g1-1} \times {}_{n-1-(k-1)(s_1+s_2)-h}C_{m-1-v-g_1} \tag{5.58}$$

and

$${}_{(k-1)s_2}C_v \times {}_{h+(k-1)s_1-1}C_{g1-1} \times {}_{n-1-(k-1)(s_1+s_2)-h}C_{m-v-g_1}. \tag{5.59}$$

By Definition 8

$$LOW2(n,m,k) = LOW2(n-1,m-1,k) \leq LOW2(n-1,m,k)$$

and

$$UPP2(n,m,k) = UPP2(n-1,m,k) \geq UPP2(n-1,m-1,k).$$

$(a.1)$. Suppose that $LOW2(n,m,k) = LOW2(n-1,m-1,k) = LOW2(n-1,m,k)$ and $UPP2(n,m,k) = UPP2(n-1,m,k) = UPP2(n-1,m-1,k)$. For a non-negative integer v such that $LOW2(n,m,k) \leq v \leq UPP2(n,m,k)$, by Lemma 2 (5.57) is equal to the sum of (5.58) and (5.59), and hence (5.54) is the sum of (5.55) and (5.56).
$(a.2)$. Suppose that $LOW2(n,m,k) = LOW2(n-1,m-1,k) = m-n+(k-1)(s_1+s_2)-g_1+h < m+1-n+(k-1)(s_1+s_2)-g_1+h = LOW2(n-1,m,k)$ or $UPP2(n,m,k) = UPP2(n-1,m,k) = m-g_1 > m-1-g_1 = UPP2(n-1,m-1,k)$. Let v be a non-negative integer v such that $LOW2(n-1,m) \leq v \leq UPP2(n-1,m-1,k)$. Then by Lemma 2 (5.57) is equal to the sum of (5.58) and (5.59). Let $v = LOW2(n,m,k) = LOW2(n-1,m-1,k) = m-n+(k-1)(s_1+s_2)-g_1+h$. Then ${}_{n-(k-1)(s_1+s_2)-h}C_{m-v-g_1} = {}_{n-(k-1)(s_1+s_2)-h}C_{n-(k-1)(s_1+s_2)-h} = 1$.
Similarly, ${}_{n-1-(k-1)(s_1+s_2)-h}C_{m-1-v-g_1} = {}_{n-1-(k-1)(s_1+s_2)-h}C_{n-1-(k-1)(s_1+s_2)-h} = 1$.
Therefore (5.57) is equal to (5.58).
Let $v = UPP2(n,m,k) = UPP2(n-1,m,k) = m-g_1$. Then, we have ${}_{n-(k-1)(s_1+s_2)-h}C_{m-v-g_1} = {}_{n-(k-1)(s_1+s_2)-h}C_0 = 1$ and ${}_{n-1-(k-1)(s_1+s_2)-h}C_{m-v-g_1} = {}_{n-1-(k-1)(s_1+s_2)-h}C_0 = 1$.
Therefore we have (5.57) is equal to (5.59).
(b) Suppose that $UPP1(n,m) = UPP1(n-1,m-1) > UPP1(n-1,m)$. By (5.52) we have $\lfloor\frac{n-h}{s_1+s_2}\rfloor = \lfloor\frac{n-1-h}{s_1+s_2}\rfloor$, and hence

$$\lfloor\frac{n-h-1}{s_1+s_2}\rfloor + 1 \geq \lfloor\frac{n-m+g_1-h}{s_1}\rfloor + 1 = UPP1(n,m) > UPP1(n-1,m) = \lfloor\frac{n-1-m+g_1-h}{s_1}\rfloor + 1. \tag{5.60}$$

Let $k = UPP1(n,m) = UPP1(n-1,m-1)$. Then by (5.60) $\frac{n-m+g_1-h}{s_1} = k-1$, and hence

$$n - m + g_1 - h = (k-1)s_1. \tag{5.61}$$

Then by (5.60) we have $\frac{n-h-1}{s_1+s_2} \geq k-1$, and we have

$$n - h - 1 \geq (k-1)(s_1 + s_2). \tag{5.62}$$

By (5.61) and (5.62)

$$m - g_1 - 1 \geq s_2(k-1). \tag{5.63}$$

By (5.48) we have $g_2 - 1 \geq m - g_1$, and hence by (5.63)

$$UPP2(n-1, m-1, k) = UPP2(n,m,k) = (k-1)s_2. \tag{5.64}$$

By (5.18) and (5.61)

$$(k-1)s_2 = m - n + (k-1)(s_1+s_2) - g_1 + h \leq LOW2(n,m,k). \tag{5.65}$$

Let v be a natural number such that $LOW2(n-1,m-1,k) = LOW2(n,m,k) \leq v \leq UPP2(n,m,k) = UPP2(n-1,m-1,k)$. Then, by (5.64) and (5.65) we have $v = (k-1)s_2$, and hence by (5.61) $n-1-(k-1)(s_1+s_2) - h = m-1-(k-1)s_2 - g_1 = m-1-v-g_1$. Therefore (5.57) is equal to (5.58), and (5.54) is equal to (5.55) $\square$

References

[1] Matsui, H., Minematsu, D., Yamauchi T., Miyadera, R.: Pascal-like triangles and Fibonacci-like sequences, Mathematical Gazette, 2010.